A GHZ-type Proof of Bell's Theorem for A Two-particle Singlet State

A. Shafiee[1] (1,3) and M. Golshani(2,3)

1) Department of Chemistry, Kashan University,
Kashan, 87317-552930, Iran.

2) Department of Physics, Sharif University of Technology,
P.O.Box 11365-9161, Tehran, Iran.

3) Institutes for Studies in Theoretical Physics & Mathematics,
P.O.Box 19395-5531, Tehran, Iran.

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[1]E-mail: shafiee@theory.ipm.ac.ir

Abstract

For the case of two spin-$\frac{1}{2}$ particles in the singlet state, we provide a GHZ-type proof of Bell's theorem by using the idea of postselected measurements. Furthermore, we show that in spite of the low efficiency of the detectors one can derive an inequality in the case of real experiments which is violated by quantum mechanics.

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Bell's theorem [1, 2] states that for a two-particle singlet state, one cannot construct a local realistic hidden-variable (HV) theory that can reproduce all the *statistical* predictions of quantum mechanics (QM). In a multi-component system, the locality assumption means that for a given component, the value of an observable does not depend on the measurements which are performed simultaneously on any other spatially separated counterpart.

In 1989, Greenberger, Horne and Zeilinger (GHZ) argued that if QM predictions hold true for the *perfect* correlations of an entangled four-particle state, deterministic local HV theories cannot reproduce QM results [3]. By the term "deterministic", they meant that one could exactly (i.e., with a probability equal to one) assign an element of reality to a predefined value of a physical quantity.

Some other versions of the GHZ theorem [4, 5], however, show that this theorem can be taken as a synthesis of Bell's theorem and Kochen-Specker's (KS) theorem [6] and indicate that we cannot attribute preexisting values to the results of simultaneous measurements of three or more correlated particles without encountering a mathematical inconsistency. Consequently, one can interpret the GHZ theorem as a proof for the breakdown of *non-contextual* HV theories. In a non-contextual theory, it is assumed that an observable assumes values independent of the values attributed to other compatible variables simultaneously measured. For a multi-component system in which each component is in a space-like separated region, there is an obvious intersection between the non-contextuality and the locality definitions. In the GHZ theorem (or any GHZ-type proof), however, it is a valuable task to clarify where the locality assumption is introduced through a HV theory and how a contextual local HV theory can be distinguished from a non-contextual one.

The GHZ theorem provides a new test for the evaluation of concepts like locality and non-contextuality on the basis of *complete* quantum correlations. But, as we shall show,it is distinct from the Bell-type inequalities which are only violated by some statistical predictions. Nevertheless, the empirical tests of the GHZ theorem have always involved technical difficulties due to the low efficiency of detectors. For this reason, some people have preferred Bell-type inequalities for observing an experimental inconsistency [7]. To take care of the limitations due to the inefficiency of the detectors, the GHZ theorem has recently been presented in the form of inequalities [8].

There have been also some attempts to provide simple proofs of KS theorem for a system of two spin $\frac{1}{2}$ particles [4, 9]. Very recently, a GHZ-type proof of Bell's theorem with two observers has been proposed by Cabello which involves *two copies* of a two-particle singlet state (instead of just one) [10] and it can also be reformulated as a proof of KS theorem [11]. In each run of this complex experiment, a source emits simultaneously two pairs of particles, each pair in a singlet state, which are numbered as 1 and 2 as well as 3 and 4, respectively. Then, an observer makes some specified spin measurements on particles 1 and 3, while in a space-like separated region a second observer makes another specified spin measurements on particles 2 and 4. Regarding the results of these measurements, Cabello concluded that the predictions of QM for two copies of singlet states cannot be reproduced by a local realistic model .Furthermore, he explained how a real justification of his argument would be possible in an actual experiment.

Here, we want to show that a GHZ-type argument can be reformulated for a *single copy* of two entangled spin $\frac{1}{2}$ particles in a singlet state, considering the idea of *postselected* spin measurements. We shall also make explicit the distinct role of the locality condition in our argument. In this manner, we are giving a GHZ-type proof of Bell's theorem without inequalities for a singlet state in an ideal case. In addition, we provide a new way for the evaluation of the contents of *both* KS and Bell theorems for our proposed experiment in an actual case in which the detection loophole, i.e. the assumption of fair sampling, seems unavoidable [12]. (The detection loophole implies that in real experiments only a fraction of the particle pairs are detected and the registered pairs are necessarily a fair sample of all pairs emitted.) Here, we avoid this assumption by choosing a subset of the detected results and, then, we show that our final remarks do not depend on which results are selected.

To begin with, we first express the GHZ theorem as follows. Suppose, we have three entangled spin $\frac{1}{2}$ particles in the following state [13],

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle_1|+\rangle_2|+\rangle_3 - |-\rangle_1|-\rangle_2|-\rangle_3\right] \tag{1}$$

where $|+\rangle_i$ or $|-\rangle_i$ indicates that the ith particle $(i = 1, 2, 3)$ has a spin up or down along the z-axis, respectively. Now, we consider the following four operators $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$ which have (1) as an eigenstate:

$$\widehat{A} = \widehat{\sigma}_{1x}\widehat{\sigma}_{2y}\widehat{\sigma}_{3y}; \quad \widehat{B} = \widehat{\sigma}_{1y}\widehat{\sigma}_{2x}\widehat{\sigma}_{3y} \tag{2}$$

$$\widehat{C} = \widehat{\sigma}_{1y}\widehat{\sigma}_{2y}\widehat{\sigma}_{3x}; \quad \widehat{D} = \widehat{\sigma}_{1x}\widehat{\sigma}_{2x}\widehat{\sigma}_{3x} \tag{3}$$

Here, $\widehat{\sigma}_{ij}$ represents the Pauli spin operators of the ith particle along the jth axis $(j = x, y)$. The expectation values of the observables A, B, C and D- corresponding to the operators $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$, respectively- in the state (1) are:

$$\langle A\rangle = \langle B\rangle = \langle C\rangle = 1; \quad \langle D\rangle = -1$$

Using a HV theory, we assume that λ is a collection of HVs which belongs to the space of states Λ ($\lambda \in \Lambda$). According to QM the result of measuring an observable is an eigenvalue of the corresponding Hermitian operator. Furthermore, for a set of the compatible observables, it requires that the allowed results of a simultaneous measurement of them must be a set of the simultaneous eigenvalues [14]. These are the constraints that QM imposes on the values of observables at the HV level, independent of the state $|GHZ\rangle$ in (1). Then, for three compatible observables A, B, and C, the average value of their product ABC should be equal to

$$E_\lambda(ABC) = \upsilon_\lambda(A)\upsilon'_\lambda(B)\upsilon''_\lambda(C) \tag{4}$$

where $\upsilon_\lambda(A)$, $\upsilon'_\lambda(B)$ and $\upsilon''_\lambda(C)$ are, respectively, the values of three compatible observables A, B, and C. QM requires that they should be all equal to 1, but because of the experimental limitations, each observable should be measured in a different experimental setup and the different symbols for each value refer to such a distinction. If we assume locality, the relation (4) changes to

$$\begin{aligned} E_\lambda(ABC) &= [\upsilon_\lambda(\sigma_{1x})\upsilon_\lambda(\sigma_{2y})\upsilon_\lambda(\sigma_{3y})]\,[\upsilon'_\lambda(\sigma_{1y})\upsilon'_\lambda(\sigma_{2x})\upsilon'_\lambda(\sigma_{3y})] \\ &\quad \times [\upsilon''_\lambda(\sigma_{1y})\upsilon''_\lambda(\sigma_{2y})\upsilon''_\lambda(\sigma_{3x})] \end{aligned} \tag{5}$$

where, $\upsilon_\lambda(\sigma_{ij})$, $\upsilon'_\lambda(\sigma_{ij})$ or $\upsilon''_\lambda(\sigma_{ij})$ represents the value of the spin component of the ith particle along the jth axis in a particular experimental arrangement. They are all supposed to be ± 1. These values are not generally assumed to be equal for the same i and j, because they may depend on what other spin components are *locally* measured along with. This is the context dependence which in QM is a consequence of the entangled form of the state $|GHZ\rangle$ in (1). The relation (1) implies that the values that can be assigned to the spin components of each particle depend on the values which are attributed to the other particles at the same time. But, if one assumes that a predefined value of any spin component of a particle is *only* determined by λ, the relation (5) changes to the following result

$$E_\lambda(ABC) = [\upsilon_\lambda(\sigma_{1x})\upsilon_\lambda(\sigma_{2x})\upsilon_\lambda(\sigma_{3x})]\left[\upsilon^2_\lambda(\sigma_{1y})\upsilon^2_\lambda(\sigma_{2y})\upsilon^2_\lambda(\sigma_{3y})\right] \tag{6}$$

which, in turn, is equal to

$$E_\lambda(ABC) = [\upsilon_\lambda(\sigma_{1x})\upsilon_\lambda(\sigma_{2x})\upsilon_\lambda(\sigma_{3x})] = \upsilon_\lambda(D) \tag{7}$$

But, on the basis of (4), we have $E_\lambda(ABC) = 1$, whereas on the basis of (7) we should have $E_\lambda(ABC) = -1$. This shows that a non-contextual HV theory cannot reproduce the predictions of QM for the state $|GHZ\rangle$ in (1). The GHZ theorem can be applied to the case of four entangled particles too. In all of these cases, the GHZ theorem implies that either spatially separated particles have influence on each other (i.e. the relation (5) and consequently (6) are incorrect) or that one cannot attribute a value to a spin component of a particle independent of the state preparation of the system (as $|GHZ\rangle$ in (1)) and that λ is not enough for the determination of these values (i.e. the relation (5) may be correct but (6) is incorrect). In either case, we are encountering the concept of contextuality, but our conclusion in the former case is a stronger one, because that leads to a contextual non-local structure for HVs.

Now, we consider the Bohmian version [15] of EPR theorem [16] and we want to show how a GHZ-type argument can be traced out here for some certain postselected spin measurements. We are dealing with two entangled spin $\frac{1}{2}$ particles in the singlet state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right] \tag{8}$$

where the kets $|+\rangle_k$ and $|-\rangle_k$ $(k = 1, 2)$ are defined similar to (1). We consider an ideal Bell experiment where a source produces a two-particle singlet state at a time. Subsequently, we introduce the following observables:

$$R = \sigma_{1x}\sigma_{2x};\ R' = \sigma_{1y}\sigma_{2y};\ Q = \sigma_{1x}\sigma_{2y};\ Q' = \sigma_{1y}\sigma_{2x} \tag{9}$$

and we define for their corresponding operators,

$$\widehat{R}\ \widehat{R'} = \widehat{S}; \quad \widehat{Q}\ \widehat{Q'} = \widehat{T};$$

The observables R, R', Q and Q' are measured independently (as is the case, e.g., in all Bell experiments) and the results of the measurements are shown in the following form:

$$r = r_{1x}r_{2x};\ r' = r_{1y}r_{2y};\ q = q_{1x}q_{2y};\ q' = q_{1y}q_{2x} \tag{10}$$

where r_{kj} and q_{kj} represent the corresponding results of the measurements of the jth spin component of the kth particle in a purposed experiment. Since r and r' should be -1 in the ideal Bell experiments for a singlet state, we always have $rr' = +1$ for the observable RR'. But, the results of two ideal measurements of Q and Q' do not lead necessarily to $qq' = -1$. Because, q and q' may accept ± 1 values independent of each other. Nevertheless, we can always *choose* those measurements in which the measurements of Q and Q' correspond to $qq' = -1$. This means that we are dealing with those measurements for which the results $q = +1$ or $q = -1$ lead to $q' = -1$ or $q' = +1$, respectively.

In this approach, the results of the measurements of Q' have the required correlation with those of Q, and we are dealing with situations for which $\langle RR' \rangle = +1$ and $\langle QQ' \rangle_{sel} = -1$ are both satisfied and we have

$$\langle ST \rangle_{sel} = rr'qq' = -1 \tag{11}$$

where the subscript *sel* indicates that only a subset of the results are selected. On the other hand, the prediction of a HV theory for the aforementioned ideal Bell experiments is

$$\begin{aligned} E_\lambda(ST) &= \upsilon_\lambda(S)\upsilon'_\lambda(T) = \upsilon_\lambda(RR')\upsilon'_\lambda(QQ') \\ &= \upsilon_\lambda(R)\upsilon_\lambda(R')\upsilon'_\lambda(Q)\upsilon'_\lambda(Q') \end{aligned} \tag{12}$$

According to (11) , QM requires $E_\lambda(ST)$ to be equal to -1 for the selected results.

Assuming locality for the spatially separated measurements in each of the systems, we have

$$\begin{aligned} E_\lambda(ST) &= [\upsilon_\lambda(\sigma_{1x})\upsilon_\lambda(\sigma_{2x})]\,[\upsilon_\lambda(\sigma_{1y})\upsilon_\lambda(\sigma_{2y})]\,[\upsilon'_\lambda(\sigma_{1x})\upsilon'_\lambda(\sigma_{2y})] \\ &\quad \times [\upsilon'_\lambda(\sigma_{1y})\upsilon'_\lambda(\sigma_{2x})] \end{aligned} \tag{13}$$

Just like the relation (5), here, we first suppose that $\upsilon_\lambda(\sigma_{ij}) \neq \upsilon'_\lambda(\sigma_{ij})$ for the same i and j. However, if we assume that λ determines a preexisting value for any spin component in a non-contextual HV theory, we would have $\upsilon_\lambda(\sigma_{ij}) = \upsilon'_\lambda(\sigma_{ij})$. One, then, obtains

$$E_\lambda(ST) = \left[\upsilon^2_\lambda(\sigma_{1x})\upsilon^2_\lambda(\sigma_{2x})\upsilon^2_\lambda(\sigma_{1y})\upsilon^2_\lambda(\sigma_{2y})\right] = 1 \tag{14}$$

This result, which plays an important role in our discussion, is in contradiction with the QM result $\langle ST \rangle_{sel} = -1$. The predictions of a non-contextual HV theory are *independent* of the selected values of q and q' and always lead to (14) for $E_\lambda(ST)$. This is because in these theories the spin state of each particle is uniquely determined by λ and we can assign a value to any spin component in an *a priori* fashion. So, the conjunction of KS and Bell theorems for two entangled spin $\frac{1}{2}$ particles leads to the same conclusion as was shown in this paper for three particles in a GHZ state.

A statistical correlation of the performed measurements on R, R', Q and Q' leads in an ideal case to the CHSH inequality [17],

$$|\langle R\rangle + \langle R'\rangle + \langle Q\rangle - \langle Q'\rangle| \leq 2 \tag{15}$$

where, $\langle R\rangle = \langle R'\rangle = -1$ and $\langle Q\rangle = \langle Q'\rangle = 0$. It is interesting that the statistical results for the measurements of R, R', Q and Q' always satisfy (15), but there are individual cases where we have $rr' = +1$ and $qq' = -1$ and therefore (11) is satisfied, which implies the refutation of the non-contextual HV theories. This shows that a GHZ-type proof for two entangled spin $\frac{1}{2}$ particles has stronger implications than the Bell theorem, though their presuppositions are similar.

So far we have considered an ideal Bell experiment where there is complete correlation and efficient detectors. In the real experiments, where we are dealing with the possibility of non-detection, the values of the spin components of each particle are taken to be $+1$, -1 and 0, where zero refers to the lack of detection. In QM, we have

$$\langle RR'\rangle_{\text{exp}} = \sum_{r,r'=\pm1} rr' P_{\text{exp}}(R = r, R' = r') \tag{16}$$

and

$$\langle QQ'\rangle_{\text{exp}} = \sum_{q,q'=\pm1} qq' P_{\text{exp}}(Q = q, Q' = q') \tag{17}$$

where P_{exp} is the joint probability of getting some results in two different experiments. In addition to the inefficiency of the detectors, the degree of correlation between the propagated particles from the initial source has a special effect. If this degree of correlation is not perfect, r and r' are not necessarily -1, even with the perfect efficiency of the detectors. Nevertheless, one can always *choose* the results in experiments where the degree of correlation is near to 1, but the low efficiency of the detectors remains the main problem.

Considering the results in the case of a nearly complete correlation, (16) would be replaced by

$$\begin{aligned}\langle RR'\rangle_{\text{exp},sel} &= P_{\text{exp}}(R = -1, R' = -1) \\ &= P_{\text{exp},sel}(R = -1) P_{\text{exp},sel}(R' = -1) \end{aligned}\tag{18}$$

where the probability measures are defined in a subset of the selected results. Here, we have assumed the statistical independence of the outcomes of the two independent measurements of R and R'. For the calculation of $\langle QQ'\rangle_{\exp,sel}$ we consider those measurements for which $qq' = -1$ or 0 and we neglect those having $qq' = +1$. Then, according to (17), we have

$$\langle QQ'\rangle_{\exp,sel} = -P_{\exp,sel}(T = -1)$$

where,

$$\begin{aligned} P_{\exp,sel}(T &= -1) = P_{\exp,sel}(Q = -1, Q' = +1) \\ &\qquad + P_{\exp,sel}(Q = +1, Q' = -1) \end{aligned} \tag{19}$$

The result $Q = -1$ is related to the case, where $q_{1x}q_{2y} = -1$ and the result $Q = +1$ refers to the case where $q_{1x}q_{2y} = +1$. A similar comment holds for Q'. In (19), the results for Q' are chosen on the basis of the result for Q, so that the conditional probabilities $P_{\exp,sel}(Q' = +1|Q = +1)$ and $P_{\exp,sel}(Q' = -1|Q = -1)$ are zero. Note that we do not eliminate the probabilities related to the lack of detection, but these probabilities do not enter (19) and have an indirect effect. Under these conditions, one can show that

$$\langle ST\rangle_{\exp,sel} = \langle RR'\rangle_{\exp,sel}\ \langle QQ'\rangle_{\exp,sel}$$

which in turn is equal to

$$\langle ST\rangle_{\exp,sel} = -P_{\exp,sel}(R = -1)P_{\exp,sel}(R' = -1)P_{\exp,sel}(T = -1) \tag{20}$$

where $P_{\exp,sel}(T = -1)$ is defined in (19). For normalized probabilities in (20), we have $-1 \leq \langle ST\rangle_{\exp,sel} \leq 0$. Now, consider the upper limit

$$\langle ST\rangle_{\exp,sel} \leq 0 \tag{21}$$

A non-contextual HV theory predicts the following relation for the average value of ST in a real experiment

$$E_\lambda(ST)_{\exp} = \sum_{\upsilon_\lambda(S),\upsilon_\lambda(T)=\pm 1} \upsilon_\lambda(S)\upsilon_\lambda(T)\ p_\lambda(\upsilon_\lambda(S), \upsilon_\lambda(T)) \tag{22}$$

where p_λ indicates probability at the level of non-contextual HVs, and

$$\upsilon_\lambda(S) = \prod_{k=1,2}\prod_{j=x,y} \upsilon_\lambda(\sigma_{kj}) = \upsilon_\lambda(T) \tag{23}$$

On the other hand, for $\upsilon_\lambda(\sigma_{kj}) = \pm 1$ we have

$$v_\lambda(S)v_\lambda(T) = 1 \tag{24}$$

The value zero does not enter (22). Using this relation, (22) reduces to

$$\begin{aligned} E_\lambda(ST)_{\text{exp}} &= p_\lambda\left[v_\lambda(S) = +1, v_\lambda(T) = +1\right] \\ &\quad + p_\lambda\left[v_\lambda(S) = -1, v_\lambda(T) = -1\right] \end{aligned} \tag{25}$$

This is a positive definite quantity, i.e.

$$E_\lambda(ST)_{\text{exp}} \geq 0 \tag{26}$$

The relation (26) was obtained independent of the assumptions which were used for deriving (21) for the selected measurements. But, if one introduces those assumptions in (26), the above inequality would be minimized at zero and we get

$$E_\lambda(ST)_{\text{exp},sel} = 0 \tag{27}$$

The relations (27) and the inequality (21) overlap only at the point zero. If we assume that the overall efficiency for detecting each of the particles in each of the systems is the same and equal to η, then $\langle ST\rangle_{\text{exp},sel}$ in (20) is of the order $-(\eta^2)^4$. Considering the fact that $\eta \ll 1$, QM predicts a very small but non-zero value for $\langle ST\rangle_{\text{exp},sel}$. Since in the experiments for each system, η^2 is detectable, the small value of the order of $-(\eta^2)^4$ is inferable. Thus, any negative result, though very small, leads to the breakdown of the relation (26).

To sum up, we have shown that a GHZ-type proof of Bell's theorem can be reformulated for a single copy of a two-particle singlet state, using some specified postselected measurements in the ideal Bell experiments. In the actual experiments, one can avoid the detection loophole by using the same idea of postselected measurements. Consequently, we derived an inequality (the relation (21))which can be really tested and show the inconsistency of the predictions of QM with those of the non-contextual HV theories.

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